ARC-LENGTH INTEGRATION & DETERMINATION OF PERIMETER OF EARTH'S ORBITAL ELLIPSE

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Determine the perimeter of Earth's orbital ellipse with semimajor axis a = 1.0 Astronomical Unit (AU) and with orbital eccentricity ecc = 0.016711 (see Reference [1]).
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$a := 1.0$ $\qquad$ $ecc := 0.016711$

$p := a \cdot \left(1 - ecc^2\right)$ $\qquad$ $r(\upsilon) := \dfrac{p}{1 + ecc \cdot \cos(\upsilon)}$

We define below a true anomaly function, υ_1, that is a function of time t in mean solar days. We set up to sweep out a radius vector for 30 time points per orbit. To do this we calculate the orbital period P via Kepler's Third Law.

$K := 0.01720209895$

Gaussian gravity constant for Sun-Earth system, in $AU^{3/2}$ per mean solar day.

$$P := \frac{2 \cdot \pi}{K} \cdot a^{\frac{3}{2}}$$

(P^2 is proportional to a^3, and the constant of proportionality is $4\pi^2/K^2$. This is another way of stating Kepler's Third Law.)

$P = 365.2568983263$

The orbital period is 365.2568983 mean solar days, to ten significant figures.

$$\upsilon_1(t) := \left\| \begin{array}{l} n \leftarrow K \cdot a^{\frac{-3}{2}} \\ M \leftarrow n \cdot t \\ E \leftarrow M \\ E \leftarrow \mathbf{root}\left(E - ecc \cdot \sin(E) - M, E\right) \\ E + 2 \cdot \mathrm{atan}\left(\dfrac{ecc \cdot \sin(E)}{1 + \sqrt{1 - ecc^2} - ecc \cdot \cos(E)}\right) \end{array} \right.$$

Define a Mathcad function that determines true anomaly as function of time. (See [2], p. 56.)

Set up and invoke Mathcad's numerical integrator of systems of differential equations, **Rkadapt** (see [4]), based upon adaptive Runge-Kutta numerical integration.

$t := 0$ Specity values of (1) dx/dt, (2) dy/dt, and (3) speed at time t = 0:

(1) $\frac{-K}{\sqrt{p}} \cdot \sin\left(\upsilon_1(0)\right) = 0$ (2) $\frac{K}{\sqrt{p}} \cdot \left(ecc + \cos\left(\upsilon_1(0)\right)\right) = 0.01749200578$

(3) $\sqrt{\left(\frac{-K}{\sqrt{p}} \cdot \sin\left(\upsilon_1(t)\right)\right)^2 + \left(\frac{K}{\sqrt{p}} \cdot \left(ecc + \cos\left(\upsilon_1(t)\right)\right)\right)^2} = 0.01749200578$

$$D(t, Y) := \begin{bmatrix} \frac{-K}{\sqrt{p}} \cdot \sin\left(\upsilon_1(t)\right) \\ \frac{K}{\sqrt{p}} \cdot \left(ecc + \cos\left(\upsilon_1(t)\right)\right) \\ \sqrt{\left(\frac{-K}{\sqrt{p}} \cdot \sin\left(\upsilon_1(t)\right)\right)^2 + \left(\frac{K}{\sqrt{p}} \cdot \left(ecc + \cos\left(\upsilon_1(t)\right)\right)\right)^2} \end{bmatrix}$$

Specify derivative function of t and 3-vector Y of derivatives (1), (2), and (3)

$$Z := \text{Rkadapt}\left(\begin{bmatrix} 0 \\ 0.01749200578 \\ 0.01749200578 \end{bmatrix}, 0, P, 30, D\right)$$

Invoke Rkadapt now and print out results below.

Mathcad's **Rkadapt** here integrates dx/dt, dy/dt, and speed = sqrt[$(dx/dt)^2$ + $(dy/dt)^2$] over one orbital period P, i.e., starting at t = 0 and stopping at t = P.

This is because the integral of the speed with respect to time is the arc length traveled. So for one orbital period, the arc length traveled is the perimeter.

Rkadapt results:

First column is time in days

Second column is dx/dt in AU/day

Third column is dy/dt in AU/day

Fourth column is arc length traveled in AU

$Z=$

0	0	0.01749201	0.01749201
12.17522994	−0.02259287	0.22882782	0.23043435
24.35045989	−0.08928365	0.43046669	0.44321722
36.52568983	−0.19687535	0.61321985	0.65568916
48.70091978	−0.34025442	0.76887611	0.8677142
60.87614972	−0.51269029	0.89059928	1.07917825
73.05137967	−0.70621354	0.97322961	1.28999417
85.22660961	−0.91204029	1.01347694	1.50010517
97.40183955	−1.12101064	1.0100046	1.70948653
109.5770695	−1.32401346	0.96341237	1.91814577
121.75229944	−1.5123762	0.87613319	2.12612136
133.92752939	−1.67820607	0.75226117	2.33348015
146.10275933	−1.81467514	0.59732862	2.54031397
158.27798927	−1.91624791	0.4180478	2.74673533
170.45321922	−1.97885263	0.22203011	2.95287269
182.62844916	−2	0.01749201	3.15886532
194.80367911	−1.97885263	−0.1870461	3.36485795
206.97890905	−1.91624791	−0.38306379	3.57099531
219.154139	−1.81467514	−0.5623446	3.77741667
231.32936894	−1.67820607	−0.71727715	3.98425049
243.50459888	−1.5123762	−0.84114917	4.19160928
255.67982883	−1.32401346	−0.92842836	4.39958486
267.85505877	−1.12101064	−0.97502059	4.60824411
280.03028872	−0.91204029	−0.97849293	4.81762547
292.20551866	−0.70621354	−0.93824559	5.02773647
304.38074861	−0.51269029	−0.85561526	5.23855239
316.55597855	−0.34025442	−0.73389209	5.45001644
328.73120849	−0.19687535	−0.57823584	5.66204148
340.90643844	−0.08928365	−0.39548268	5.87451342
353.08166838	−0.02259287	−0.19384381	6.08729629
365.25689833	$3.17368161 \cdot 10^{-9}$	0.01749201	6.30023863

$2 \cdot \pi = 6.2831853072$ $\qquad \dfrac{Z_{31,4}}{2 \cdot \pi} = 1.0027141212$

(For a perfect circle, the quotient should be 1.0.)

According to p. 12 of my textbook [2], one AU, in km, is

$$23454.80191 \cdot 6378.135 = 149597892.980$$

So the conversion factor KmPerAU is given by this:

$$KmPerAU := 149597892.980$$

And the arc length Earth travels in one orbital period (the perimeter of the orbital ellipse), in kilometers, is

$$Z_{31,4} \cdot KmPerAU = 942502424.815$$

That is, 942,502,424.815 km.

Alison Klesman, Senior Editor at *Astronomy m*agazine*,* gives 940 million km as the approximate perimeter of Earth's orbital ellipse [3], due to a formula attributed to the Indian mathematical genius Ramanujan. So Dr. Klesman's analysis and Ramanujan's formula are quite good to estimate the perimeter of Earth's orbital ellipse!

REFERENCES

[1] Sean E. Urban and P. Kenneth Seidelmann, Editors, *Explanatory Supplement to the Astronomical Almanac* (3rd Edition, 2013) Uniiversity Science Books, Mill Valley, California, p. 338.

[2] Roger L. Mansfield, *Topics in Astrodynamics* (2003), Astronomical Data Service, Colorado Springs, Colorado. (See http://astrotopics.astroger.com.) The formula for true anomaly $\upsilon(t)$ is given on p. 56 of my book. The formulae for dx/dt and dy/dt are given on p. 60.

[3] Alison Klesman (Senior Editor), Ask Astro, *Astronomy Magazine,* November 2024, p. 53, "How Big is the Ellipse that Our Planet Travels in a Year around the Sun?" (Question asked of Ask Astro by J.J. Muedespacher, Mexico City, Mexico).

[4] Mathcad Prime 10, published by PTC. See https://www.mathcad.com/en/ and http://mathcadwork.astroger.com.

A FINAL NOTE

The constant *KmPerAU* above is based upon values given in my book on p. 12, which values were determined for Space-Track Earth Model 68, Revised (STEM 68R). Such astronomical constants are updated from time to time by such authorities as JPL, USNO, NORAD, the USSF, and the International Astronomical Union (IAU). Earth's orbital ellipse perimeter 942,502,424.815 km is given down to the meter just because Mathcad can do that. *Alison Klesman's value of 940 million km is the value to remember.*